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On The Relaxation Of Infinite Range Spin Glasses

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Abstract. The relaxation of the Sherrington–Kirkpatrick model for spin glasses is studied at low temperatures. A recently developed numerical method is used by which the infinitely large system is simulated, hence finite size effects are avoided. For parallel dynamics the decay of the energy as well as of the magnetization is investigated. For low temperatures we find evidence for a relaxation into a state which is characterized by non-equilibrium values of the energy and a non-vanishing remanent magnetization.
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In thermal equilibrium the infinite range spin glass (the Sherrington Kirkpatrick (SK) model [1]) is characterized by infinite energy barriers and non-ergodic behaviour below the critical temperature [2]. But in a typical experiment the system is prepared in a state far from equilibrium and relaxes by thermal noise and decay of energy. At zero temperature the dynamics always decreases the energy and therefore gets trapped in metastable states [3]; hence the system decays to a state with an energy much higher than the one of the ground state and with a nonzero remanent magnetization [4, 5]. For nonzero temperatures, however, the scenario is less clear. There exists still an exponentially large number of metastable states with energy values much higher than the equilibrium value [7]. But to our knowledge it is not known whether these metastable states are separated by infinitely high energy barriers from the equilibrium configuration and whether the system actually gets trapped in such states. Hence, the question remains open whether at nonzero temperatures the system approaches a state with nonzero remanent magnetization and higher energy than in thermal equilibrium. While previous numerical results [5, 6] have indicated that the SK model decays to thermal equilibrium, recent simulations have shown that some properties of the model strongly depend on the size of the system [4]. Therefore we use the method of Eißfeller and Opper [9] to simulate the infinitely large system.

Note, that the infinite range Ising ferromagnet has stable states far from equilibrium with high energy and nonzero memory to the initial state. If at low temperatures a small positive magnetic field is applied, an initial state with a sufficiently large negative magnetization evolves towards a metastable state with negative magnetization and a higher energy per spin than in equilibrium. This metastable state is separated by an infinite energy barrier from thermal equilibrium. Whether this scenario extends to spin glasses with an infinite number of metastable states is subject of our present investigation.

The method proposed by Eißfeller and Opper [9] to study the (parallel) dynamics of infinitely large disordered systems combines the dynamical functional approach, which allows to perform the limit $N \to \infty$ exactly, and a Monte Carlo simulation of the resulting stochastic single particle equations. In [11] the approach is used to examine a variant of the Sherrington–Kirkpatrick model for spin glasses. The phase transition as a function of the asymmetry in the random couplings [10] is confirmed and investigated in detail. In the present paper, we extend the method to nonzero temperatures in order to investigate the temporal development of the energy as well as of the magnetization.

Independently, Ferarro [12] did the same extension to nonzero temperatures in the case of symmetric couplings, using a "trajectory scaling" (see below). We agree with his findings concerning the energy, but suggest a new interpretation of the results of the remanent magnetization, which he assumed to be zero for nonzero temperatures.

The model consists of N Ising spins $S_i = \pm 1$. Every spin S_i is connected to all other spins S_j with $i < j$ by independent Gaussian couplings J_{ij} with zero mean and variance $1/N$. The symmetry of the matrix of couplings is given by the parameter η:

$$\langle J_{ij} J_{ji} \rangle_J = \eta / N \quad , \tag{1}$$

where the brackets denote an average over the distribution of couplings. The couplings are fully antisymmetric if $\eta = -1$ and totally uncorrelated if $\eta = 0$. Symmetric couplings as in the SK model correspond to $\eta = 1$.

Instead of directly simulating the system of dynamical equations

$$S_i(t+1) = \mathrm{sign}[h_i(t)] \quad , \quad i = 1, ..., N \quad , \tag{2}$$

with the internal fields

$$h_i(t) = \sum_{j \neq i} J_{ij} S_j(t) \quad , \tag{3}$$

where all spins are updated in parallel, Eißfeller and Opper [9, 11] follow the dynamical functional approach [13]. This allows to perform the average over the random couplings J_{ij} and to transform the remaining expression using saddle point methods, which are exact in the thermodynamic limit $N \to \infty$. The result is a system of *stochastic* dynamical equations

$$\begin{aligned} S(t+1) &= \mathrm{sign}[h(t)] \\ h(t) &= \Phi(t) + \eta \sum_s K(t,s) S(s) \quad , \end{aligned} \tag{4}$$

where the correlations of the time dependent Gaussian noise variables $\Phi(t)$ and the response function $K(t,s)$ are determined by the saddle point equations as

$$\langle\Phi(s)\Phi(\tau)\rangle_\Phi = C(s,\tau) = \langle S(s)S(\tau)\rangle_\Phi \qquad (5)$$
$$K(t,s) = \langle \partial S(t)/\partial\Phi(s)\rangle_\Phi \quad .$$

For a full derivation of the dynamical single particle equations (4,5) and a detailed description of the Monte Carlo procedure used to simulate these equations see [11].

In order to include noise in the dynamical single particle equations, we add a new random variable $r(t)$ to the internal field

$$S(t+1) = \mathrm{sign}[h(t) + r(t)] \quad , \qquad (6)$$

where $r(t)$ is generated according to:

$$r(t) = \frac{1}{2\beta}\ln\left[\frac{1-x(t)}{x(t)}\right], \text{ with } x(t) \text{ equally distributed in } [0,1] \quad . \qquad (7)$$

In the following the asymmetry η is set to 1. In this case the noise parameter β can be interpreted as the inverse temperature $\beta = 1/T$ and the dynamics obeys detailed balance. In thermal equilibrium, it leads to a Gibbs distribution of the spin configurations with the partition function [14, 15]:

$$Z = \mathrm{Tr}_S \; \exp\left[\sum_i \ln\left\{2\cosh\left(\beta\sum_j J_{ij}S_j\right)\right\}\right] \quad . \qquad (8)$$

In equilibrium, the mean energy per spin of the system is given by

$$\langle E/N\rangle_\beta = -1/N \; \partial\ln Z/\partial\beta = \langle \tanh[\beta h_i] \; h_i\rangle_\beta \quad . \qquad (9)$$

Using the Monte Carlo procedure mentioned above, the time dependent energy can be calculated from

$$e(t) = E(t)/N = -\langle\tanh[\beta h(t)] \; h(t)\rangle_\Phi = -\langle\mathrm{sign}[h(t)+r(t)] \; h(t)\rangle_{\Phi,r} \; . \qquad (10)$$

In order to compare the energy of the system extrapolated to infinite times $e_\infty = \lim_{t\to\infty} e(t)$ with results of equilibrium statistical mechanics we derived the free energy of the system with the partition function (8) in replica symmetry – following Fontanari and Koeberle [16]. We found that the parallel SK model (or Little model [17]) follows the same thermodynamics as the sequential SK model, but that the free energy is twice the original one. The result confirms the (in the case of symmetric couplings) numerical findings obtained by Brunetti *et al.* [18], which show for example that the difference between the energy of the Little model with N spins and that of the SK model with $2N$ spins disappears for $N \to \infty$. Hence, the full hierarchical solution of the SK model is valid for the Little model (eq. (8)), too.

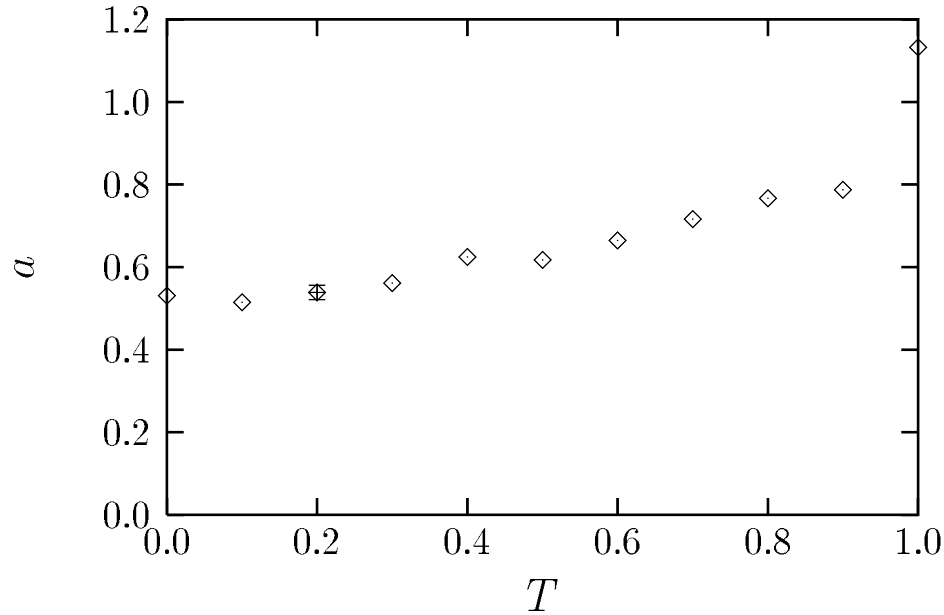

Figure 1: The temperature dependence of the exponent a. The error bar is estimated from 10 independent runs (3σ).

Using the Monte Carlo procedure described in [11] we simulate the single particle equations (6) with the internal fields from (4). The decay of the energy is calculated for the first 130 time steps for various values of the temperature. In order to estimate the averages over the random Gaussian variables Φ, occuring in the saddle point equations (5), we simulate $N_T = 10^6$ trajectories, starting from a fully magnetized state $S(t = 0) = 1$ for all trajectories. In the following all fits were done in the temporal range from $t = 20$ to $t = 130$. The results could be fitted well by the function

$$e(t) = \text{const} \times t^{-\mathrm{a}} \; + \; e_\infty \quad , \tag{11}$$

where the parameters a and e_∞ are functions of the temperature. Fig. 1 shows $a(T)$. The resulting extrapolated values e_∞ are displayed in fig. 2 together with the replica symmetric solution and the solution of the full replica symmetry breaking equations [19]. The latter was taken from Ferraro [12] (fig. 3). Surprisingly, the simulated energy e_∞ shows at low temperatures a non-monotonic behaviour, which contradicts equilibrium theory. On the other hand, the simulations are in good agreement with equilibrium theory for temperatures close to $T = 1$. Note, that for temperatures higher than $T \simeq 0.6$ there are less than exponentially many metastable states (solutions of the TAP equations) [7]. The non-monotonic behaviour of the e_∞ at low temperatures was also found by Ferraro, who assumed a linear dependence of the time dependent energy on the inverse of the number of trajectories N_T ($N_T = 8000$ and $N_T = 16000$ in [12]).

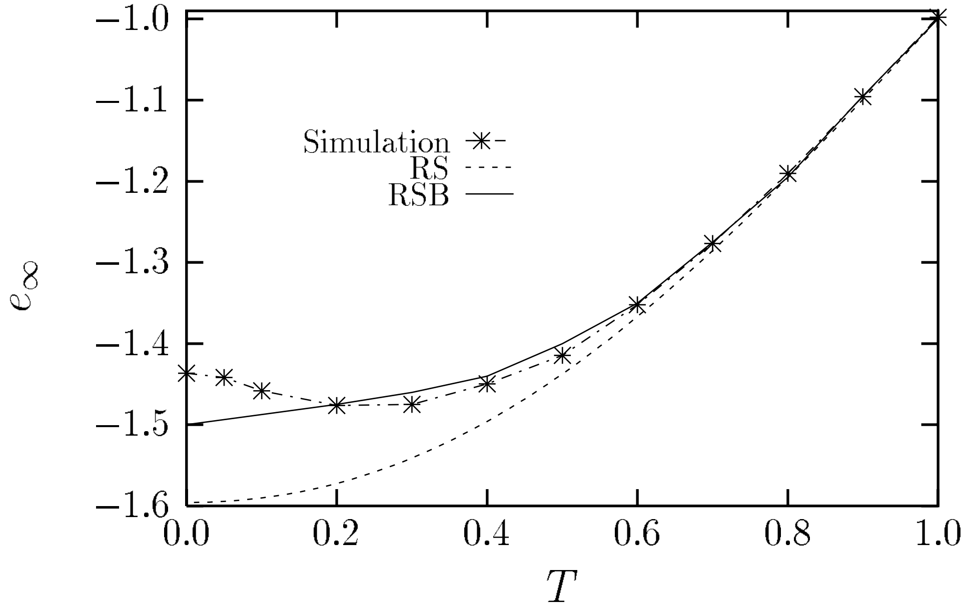

Figure 2: The extrapolated values of the energy (stars) as a function of the temperature. The error bar as a result of 10 independent runs for $T = 0.2$ is smaller than the symbol size. The dotted line represents the replica symmetric solution, whereas the solid line shows the numerical solution of the replica equations [19] and was taken from [12].

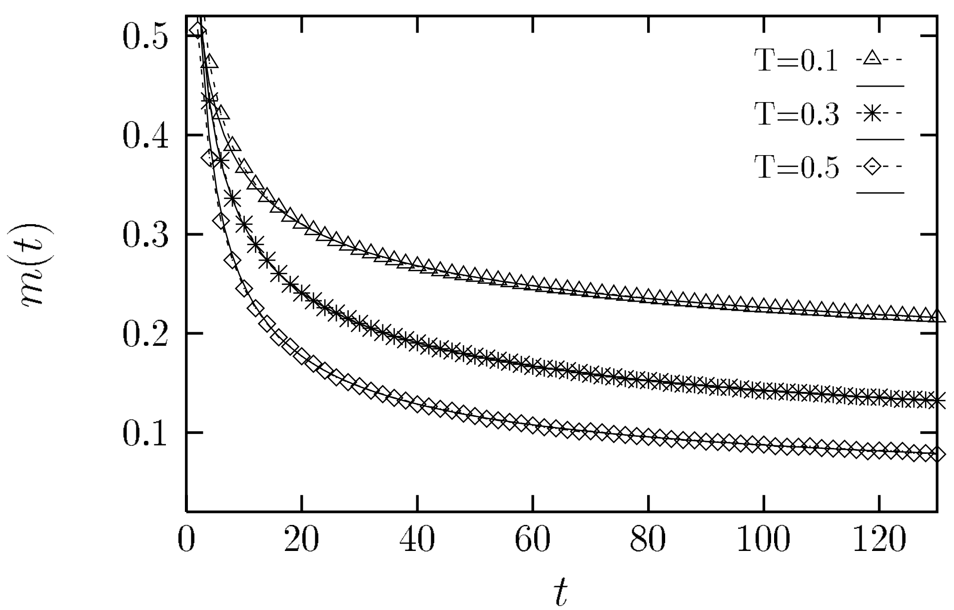

Figure 3: The decay of the magnetization for different temperatures. The solid lines display the corresponding fit functions.

The magnetization describes the system's memory of its initial state

$$m(t) = \langle S(0)S(t)\rangle_\Phi \quad . \tag{12}$$

In equilibrium the system should be completely independent of its initial configuration or in other words: the equilibrium state is characterized by a zero remanent magnetization $m_\infty = \lim_{t\to\infty} m(t) = 0$.

The decay of the magnetization for the first 130 time steps is displayed in fig. 3 for various temperatures. Note, that the magnetization vanishes at uneven times [11], which is not shown in fig. 3. The nonzero part of the magnetization reveals the following relaxation behaviour:

$$m(t) = \begin{cases} \text{const} \times t^{-\mathrm{b}} \quad (+\, m_\infty\,) & \text{for} \quad 0 < T \tilde{<} 1 \\ \text{const} \times t^{-\mathrm{b}} \exp[-t/\tau] & \text{for} \qquad T \tilde{>} 1 \end{cases} \quad , \tag{13}$$

where the parameters b, τ and m_∞ again are functions of the temperature – as displayed in fig. 4 for the exponent b. At a temperature $T \simeq 1$ we find a transition in the relaxation behaviour of the magnetization from an algebraic decay to one with an additional exponential factor, in agreement with the critical temperature $T_c = 1$ predicted by equilibrium theory. Furthermore, we find a slow decay of the remanent magnetization m_∞ to zero with increasing temperature (fig. 5). Up to the critical temperature a memory of the initial state remains.

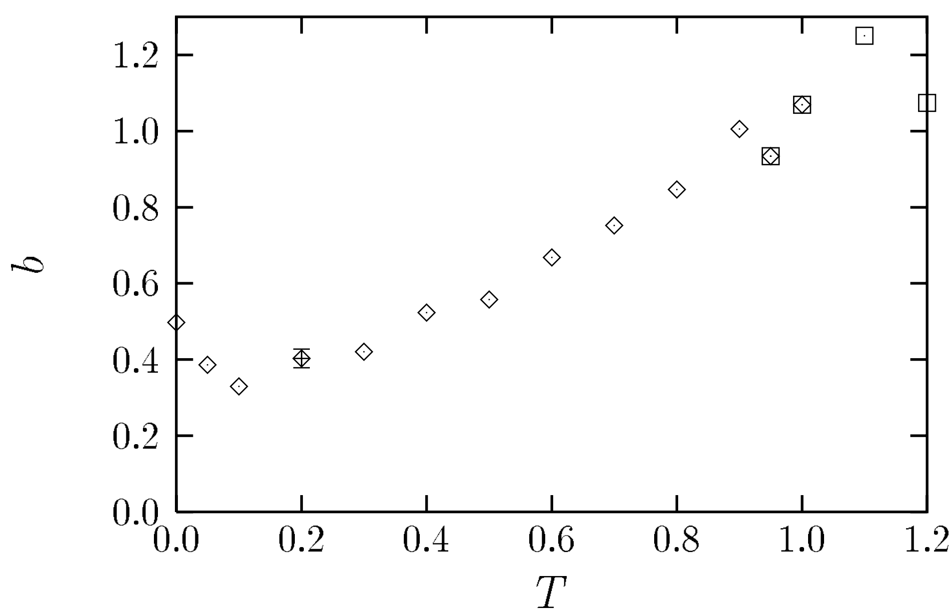

Figure 4: Temperature dependence of the exponent b. The error bar is estimated from 10 independent runs (3σ). Up to the temperature $T = 1$ the power law was fitted; for higher temperatures an additional exponential factor had to be included. At the temperatures $T = 0.95$ and $T = T_c = 1$ the exponents b for both fits coincide.

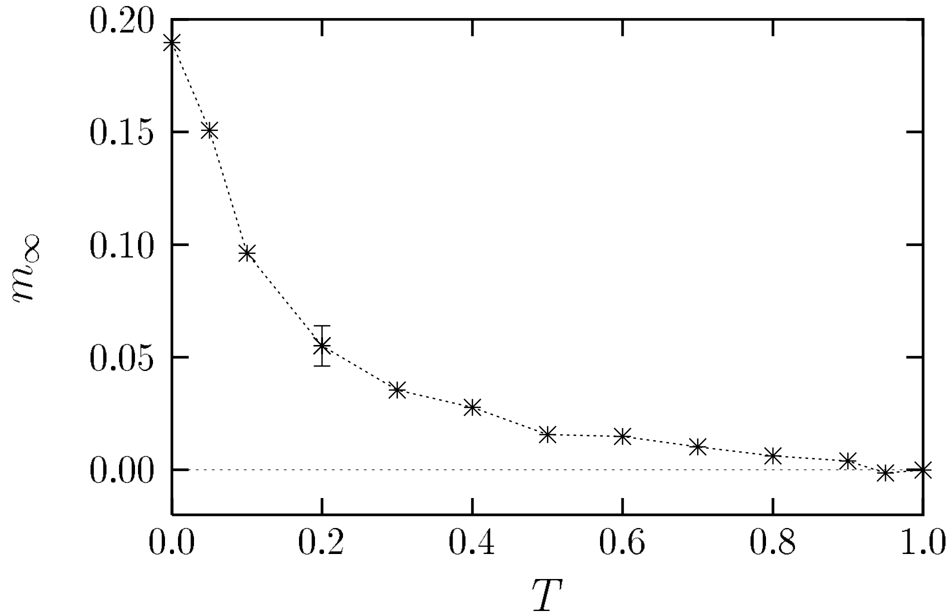

Figure 5: The remanent magnetization as a function of the temperature. The error bar is estimated from 10 independent runs (3σ).

Our simulations give further evidence for a nonzero remanent magnetization at low temperatures. The memory effect to the initial conditions at low temperatures is also reflected in the behaviour of the response function $K(t,s)$. As can be seen from eq. (5), this function describes the average response of the magnetization at time t to small variations of an external field at a previous time s. In fig. 6 the response function $K(t,t-\Delta t)$ at low temperature ($T=0.2$) is displayed as a function of the time difference $\Delta t = t-s$ for various times t and averaged over 10 independent runs. The system strongly responds to variations at times s close to time t, whereas intermediate times do not influence the system. But a significant increase of K at large Δt can be observed, indicating a stronger memory to the initial conditions. For high temperatures, memory effects are short ranged and K decays after a few time steps. Moreover, the fact that the response functions are identical at large times s for all displayed times t reflects a stationary behaviour, *i.e.* a dependence on time differences only.

We extended our investigations of the remanent magnetization at nonzero temperatures to the asymmetric case $\eta < 1$. In a region of temperatures $T \tilde{<} 1$ and values of the asymmetry parameter $\eta > 0.825$ (fig. 7) we find the same behaviour as described above, namely a non-vanishing remanent magnetization and a memory effect in the response function K.

Furthermore, we studied the dependence on the numbers of trajectories in the range $10^4 \leq N_T \leq 3.2\ 10^5$. If we assume a linear dependence of the time dependent energy on

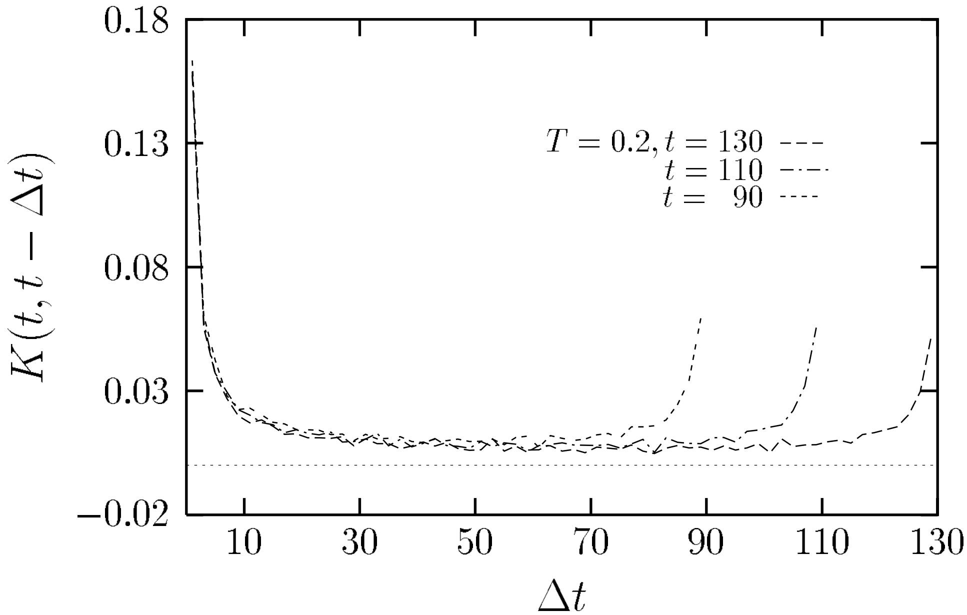

Figure 6: The response function for $T = 0.2$ and different times t as a function of the time difference $\Delta t = (t - s)$ – averaged over 10 independent runs. The fluctuations at intermediate times (due to numerical noise) have been smoothed down by this average, which accentuates the memory effect.

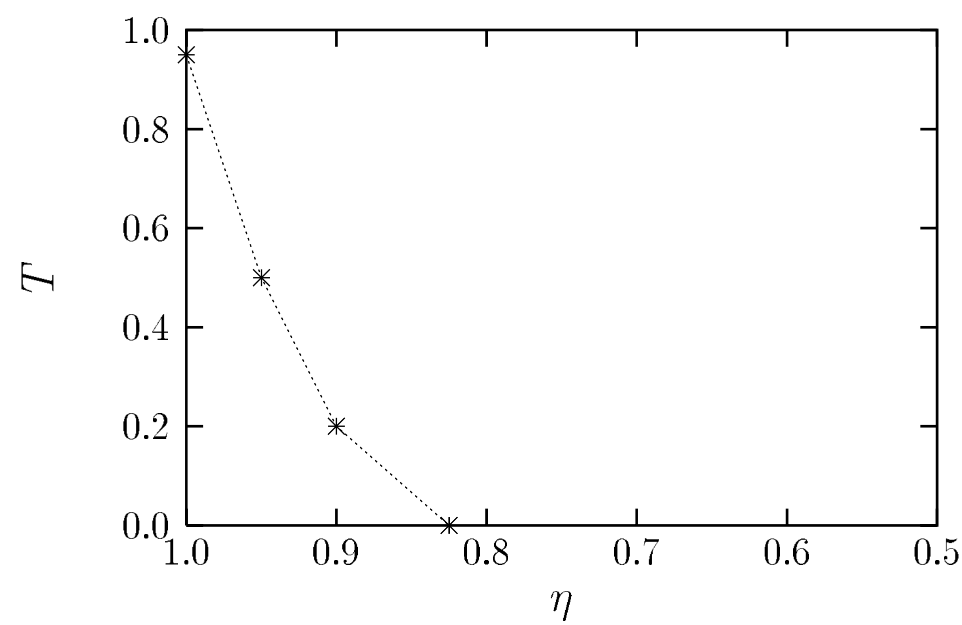

Figure 7: Phase diagram as a function of the asymmetry parameter η and the temperature T. Below the broken line the remanent magnetization is found to be different from zero $m_\infty \neq 0$.

$1/N_T$ as in [12], we cannot confirm the slope reported there. More importantly, we find clear deviations from the linear dependence for high numbers of trajectories $(3.2\ 10^5, 1.6\ 10^5)$. These deviations may be responsible for the quantitative differences concerning the values of the exponents a.

In conclusion, we have examined parallel non-equilibrium dynamics of the SK model and find evidence for a relaxation into a state characterized by a nonzero remanent magnetization and a non-equilibrium value of the energy for low but nonzero temperatures. This indicates that spin glasses relax to stable states far from thermal equilibrium.

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